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# The non-existence of desarguesian $t$ -parallelisms, $t$ an odd prime

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**Abstract.** In this article, it is shown that when  $t$  is a prime, partial Desarguesian  $t$ -parallelisms in  $PG(zt - 1, q)$  of  $m$   $t$ -spreads are equivalent to translation nets of order  $q^{zt}$  and degree  $1 + q + m(q^{zt} - q)$ . Thus, a bound is established for the number of  $t$ -spreads in partial Desarguesian  $t$ -parallelisms, when  $t$  is an odd prime. This also shows that there cannot be Desarguesian  $t$ -parallelisms when  $t$  is an odd prime.

**Keywords:**  $t$ -parallelism, translation planes, rational Pappian partial spreads

**MSC 2000 classification:** primary 51E20, secondary 51A40

## 1 Introduction.

This article considers partial Desarguesian  $t$ -parallelisms in  $PG(zt - 1, q)$ , where  $t$  is an odd prime. Recall that a  $t$ -spread of a vector space  $V$  of dimension  $zt$  over  $GF(q)$ , is a partition of the non-zero vectors of  $V$  by a set of mutually disjoint  $t$ -dimensional vector subspaces. A Desarguesian  $t$ -spread is a  $t$ -spread with the property that there exists a field  $L$  isomorphic to  $GF(q^t)$  so that  $V$  is a  $z$ -dimensional vector space over  $L$  and the elements of the  $t$ -spread are 1-dimensional  $L$ -subspaces. A Desarguesian  $t$ -parallelism is a covering of the  $t$ -dimensional subspaces by a set of  $t$ -spreads that do not share a common  $t$ -space.

There are a number of Desarguesian 2-parallelisms in  $PG(3, q)$ , and in this case, the terminology ‘regular parallelism’ is used. In this case, there must be a set  $1 + q + q^2$  2-spreads in a Desarguesian parallelism. The known Desarguesian parallelisms are follows: There are two mutually non-isomorphic parallelisms in  $PG(3, 2)$ , two in  $PG(3, 8)$ , due to Denniston [1], two in  $PG(3, 5)$ , due to Prince [5] and an infinite class in  $PG(3, q)$ , where  $q \equiv 2 \pmod{3}$ , due to Penttila and Williams [4] and this class contains all of the previously mentioned examples.

The set of Desarguesian parallelisms in  $PG(3, q)$  is equivalent to the set of translation planes of order  $q^4$  with spread in  $PG(7, q)$  covered by a set of derivable partial spreads of degree  $1 + q^2$  that mutually share a regulus partial spread of degree  $1 + q$ . Furthermore, it is precisely this connection that we wish to generalize in this article.

Our main result shows that when  $t$  is a prime, there is an equivalence with partial Desarguesian  $t$ -parallelisms in  $PG(zt - 1, q)$  of  $m$   $t$ -spreads and vector space translation nets of degree  $1 + q + m(q^t - q)$  and order  $q^{zt}$ , which consist of  $m$  rational Desarguesian partial spreads of degree  $1 + q^t$  and order  $q^{zt}$  that share a regulus net of degree  $1 + q$ . Since such translation

nets are bounded by the degree  $1 + q + \frac{(q^{zt-1}-1)}{(q^t-1-1)}(q^t - q) = 1 + q^{zt}$ , we obtain a strong bound on the cardinality of partial Desarguesian  $t$ -parallelisms, which also shows that Desarguesian  $t$ -parallelisms cannot exist with  $t$  is an odd prime.

## 2 Partial Desarguesian $t$ -parallelisms, $t$ an odd Prime.

In Jha and Johnson [2], there is a general study of the connection between Desarguesian  $t$ -spreads in  $PG(zt-1, q)$  and rational Desarguesian partial spreads of degree  $1 + q^t$  and order  $q^{zt}$  in  $PG(2zt-1, q)$ . By a ‘rational Desarguesian partial spread’, it is means that there is a field  $L$  of  $zt \times zt$  matrices isomorphic to  $GF(q^t)$ , such that the components of the partial spread may be represented in the following form:

$$x = 0, y = xM; M \in L.$$

It is also assumed that  $L$  contains a subfield  $K$ , whose elements are  $\alpha I_{zt}$ , for all  $\alpha \in GF(q)$ . This means that the partial spread is a  $K$ -regulus in  $PG(2zt-1, q)$ . We recall that a ‘ $K$ -regulus’ is a set of  $q+1$   $zt-1$  dimensional projective subspaces that are covered by a set of lines, such that a line intersecting three such subspaces intersects all  $q+1$  of the subspaces.

**Theorem 1.** (Jha and Johnson [2], (2.6)). *There is a 1-1 correspondence between Desarguesian  $t$ -spreads of a  $zt$ -dimensional vector space over  $GF(q)$  and rational Desarguesian nets of degree  $1 + q^t$  and order  $q^{zt}$  that contain a  $K$ -regulus in a vector space of dimension  $2zt$  over  $GF(q)$ .*

There is also a general correspondence between partial Desarguesian 2-parallelisms and translation planes covered by rational Desarguesian nets, also established by the authors in [2] that generalizes unpublished work of Prohaska and Walker, and is inspired by the work of Walker [6], and Lunardon [3].

**Theorem 2.** (Jha and Johnson [2] (2.8)). *Let  $V$  be a vector space of dimension  $4z$  over  $GF(q)$ , and let  $\mathcal{R}$  be a regulus of  $V$  (of  $PG(4z-1, q)$ ). Let  $\Gamma$  be a set of rational Desarguesian nets isomorphic of degree  $1 + q^2$  and order  $q^{2z}$  containing  $\mathcal{R}$ . Then*

$$\cup(\Gamma - \mathcal{R}) \cup \mathcal{R}$$

*is a partial spread if and only if for any choice of component  $A$  of  $\mathcal{R}$ , considered as a  $2z$ -dimensional  $GF(q)$ -space,  $\Gamma$  induces a partial 2-parallelism of  $A$ .*

We now generalize Theorem 2, for partial Desarguesian  $t$ -parallelisms, when  $t$  is an odd prime.

**Theorem 3.** *Let  $V$  be a vector space of dimension  $2tz$  over  $GF(q)$ , and let  $\mathcal{R}$  be a regulus of  $V$  (of  $PG(2tz-1, q)$ ). Let  $\Gamma$  be a set of rational Desarguesian nets isomorphic of degree  $1 + q^t$  and order  $q^{tz}$  containing  $\mathcal{R}$ . If  $t$  is a prime, then*

$$\cup(\Gamma - \mathcal{R}) \cup \mathcal{R}$$

*is a partial spread if and only if for any choice of component  $A$  of  $\mathcal{R}$ , considered as a  $tz$ -dimensional  $GF(q)$ -space,  $\Gamma$  induces a Desarguesian partial  $t$ -parallelism of  $A$ .*

*Proof.* Let  $A$  be a  $zt$ -dimensional vector subspace of a  $2zt$ -dimensional vector space  $V$  over a field  $GF(q)$ . Let  $S_1$  be a Desarguesian  $t$ -spread of  $A$  and by Theorem 1 form the associated rational partial  $zt$ -spread  $R_1$  of degree  $1 + q^t$  in  $V$  that contains the  $q$ -regulus partial spread  $N$  (of degree  $q+1$ ). Now take a second Desarguesian  $t$ -spread of  $A$   $S_2$  and form the rational partial  $zt$ -spread  $R_2$  of degree  $1 + q^t$  containing  $N$ . The proof of our result is finished if it could be shown

that the two rational partial spreads do not share any points outside of the regulus  $N$ . So, assume that  $Q$  is a common point of  $R_1$  and  $R_2$  that does not lie in  $N$  (on a component of  $N$ ). Since rational Desarguesian partial spreads are subplane covered nets, covered by Desarguesian subplanes of order  $q^t$ , therefore, there is a subplane  $\pi_1$  of  $R_1$  of order  $q^t$  and a subplane  $\pi_2$  of  $R_2$  of order  $q^t$  that share the point  $Q$ . Since  $\pi_1$  is a  $2t$ -dimensional  $GF(q)$ -subspace generated any two  $t$ -intersections of components of the regulus  $N$ , take  $x = 0, y = 0, y = x$  are three components of  $N$ , say taking  $A$  as  $x = 0$ . Then there are points on  $x = 0, y = 0, y = x$  in  $\pi_1$  as follows:  $P_{x=0} + P_{y=0} = Q = Z_{x=0} + Z_{y=x} = W_{y=0} + W_{y=x}$ , where the subscripts indicate what component the points are located.

Take the subspace  $\Lambda = \langle P_{x=0}, P_{y=0}, Z_{x=0}, Z_{y=x}, W_{y=0}, W_{y=x} \rangle$ , which is at least 2-dimensional over  $GF(q)$ . If the subspace is 2-dimensional over  $GF(q)$ , then since  $N$  is a regulus and since  $\Lambda$  contains points of  $x = 0, y = 0, y = x$  then  $\Lambda$  must intersect all of components of  $N$ , which means that  $Q$  cannot be in  $\Lambda$ . Therefore,  $\Lambda$  has dimension at least 3 over  $GF(q)$ . Let  $U = \pi_1 \cap \pi_2$  as a subspace of dimension at least 3. We claim that the dimension is 4. Actually, no two of the subspaces  $\langle P_{x=0}, P_{y=0} \rangle, \langle Z_{x=0}, Z_{y=x} \rangle, \langle W_{y=0}, W_{y=x} \rangle$  can be equal since otherwise a 2-dimensional subspace would non-trivially intersect three components of a regulus, and the same contradiction applies. By the construction given in [2] to establish Theorem 1, then  $\pi_1$  and  $\pi_2$  admit the group element  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ , which means that  $\pi_1 \cap \pi_2$  also admits this group.

Therefore, this implies that the intersection on  $x = 0$  is at least two dimensional. Hence,  $\Lambda$  is 4-dimensional, as it is generated by two mutually disjoint 2-dimensional  $GF(q)$ -subspaces.

Let  $P_{x=0}^* + P_{y=0}^* = Q$ , where the points are in  $\pi_2$ . Then  $P_{x=0} + P_{y=0} = P_{x=0}^* + P_{y=0}^* = Q$ , clearly implies that  $P_{x=0} = P_{x=0}^*, P_{y=0} = P_{y=0}^*$  so that  $\Lambda$  is common to  $\pi_1 \cap \pi_2$ .

Define a point-line geometry as follows: The ‘points’ are the points of  $U$ , the ‘lines’ are the lines  $PQ$ , of both  $\pi_1$  and  $\pi_2$ , where,  $P, Q$  are distinct points of  $U$ . We claim then  $U$  becomes an affine plane and then an affine subplane of both  $\pi_1$  and  $\pi_2$ . To see this, let  $P$  and  $Q$  be distinct points of  $U$  and form the unique line  $PQ$  common to both  $\pi_1$  and  $\pi_2$ . Two lines of  $U$  are parallel if and only if they are parallel in  $\pi_1$  and parallel in  $\pi_2$ . Now let  $PQ$  and  $RT$  be lines of  $U$  that are not parallel. Since they are both lines of  $\pi_1$  and lines of  $\pi_2$  then these two lines intersect in a common point of  $\pi_1$  and  $\pi_2$ . Hence, two lines of  $U$  are either parallel or intersect uniquely. Let  $\ell$  be a line  $PQ$  of  $U$  and let  $R$  be a point of  $U$  not incident with  $PQ$ . Form the line  $(P - P)(Q - P)$  of  $U$  and note that  $R - P$  is not incident with this line  $0(Q - P)$ , a common component of both  $\pi_1$  and  $\pi_2$ , since  $0$  and  $Q - P$  are in  $\pi_1 \cap \pi_2$ . So, we may assume that  $PQ$  contains  $0$ , that is, it is a common component  $\ell$  of  $\pi_1$  and  $\pi_2$ . Assume without loss of generality that  $Q$  is not  $0$ . Then  $\ell + R = R(Q + R)$  is the unique common line of  $\pi_1$  and  $\pi_2$  parallel to  $\ell$  and incident with  $R$ . Hence,  $U = \pi_1 \cap \pi_2$  is an affine translation subplane and as a subspace is of dimension at least 4. But,  $\pi_1$  is a Desarguesian affine plane of order  $q^t$  and  $U$  is an affine subplane of  $\pi_1$  of order  $q^a$ , for  $a \geq 2$ . Therefore,  $a$  must divide  $t$ . Now assume that  $t$  is an odd prime. Then,  $t = a$ , since  $t$  is an odd prime. We have then shown that  $R_1 \cup R_2$  is a net of degree  $1 + q + 2(q^t - q)$ , which is the union of two rational Desarguesian partial spreads of degrees  $1 + q^t$ , both of which share the regulus  $N$ . This completes the proof of the theorem.  $\square$

So, partial Desarguesian  $t$ -spread in a vector space of dimension  $zt$  over  $GF(q)$  of cardinality  $m$  induces a partial spread of degree

$$1 + q + m(q^t - q).$$

The maximum partial spread has total degree  $q^{zt} - q + 1 + q = 1 + q^{zt}$ . Therefore,

$$m \leq (q^{zt-1} - 1)/(q^{t-1} - 1).$$

We then have the following corollary:

**Corollary 1.** *If  $t$  is a prime, the maximum cardinality of a Desarguesian partial  $t$ -spread in a vector space of dimension  $zt$  over  $GF(q)$  is  $[(q^{zt-1} - 1)/(q^{t-1} - 1)]$  and if this bound is taken on then  $t - 1$  must divide  $zt - 1$ .*

**Theorem 4.** *Of course, if  $t = 2$ , then  $(q^{2r-1} - 1)/(q - 1)$  is the number of spreads of a 2-parallelism.*

*If  $t$  is an odd prime, then to achieve a parallelism, we require*

$$\frac{(q^{zt} - 1)(q^{zt-1} - 1) \dots (q^{zt-t+1} - 1)}{(q^t - 1)(q^{t-1} - 1) \dots (q - 1)}$$

*$t$ -spreads.*

*Therefore, a Desarguesian  $t$ -parallelism exists for  $t$  a prime if and only if  $t = 2$ .*

For example, suppose  $t = 3$  and  $z = 3$ , we then are considering Desarguesian 3-spreads in 9-dimensional vector spaces over  $GF(q)$ .

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